Convergence, cycles and complex dynamics of financing investment

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22. March 2013

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December 10, 2013

Abstract

This paper demonstrates the diverse dynamical possibilities of a simple macroeconomic model of debt-financed investment-led growth in the presence of interest rate rules. We show possibilities of convergence to steady state, growth cycles around it as well as various complex dynamics from codim 1 and codim 2 bifurcations. The effectiveness of monetary policy in the form of interest rate rules is examined under this context.

Keywords: Growth cycles, Hopf bifurcation, complex dynamics, Taylor rule.

JEL classification: C62; C69; E12; E32; E44; G01

1 Introduction

In this paper, we consider the complex dynamical possibilities of real-financial interaction in demand constrained closed economies. In particular, we investigate whether a combination of debt-financing of investment and simple monetary policy rules can provide endogenous explanations for persistent growth cycles as well as more complex dynamical possibilities emerging from such economies. In addition, we also examine the effectiveness of monetary policy, in the form of interest rate rules, in these economies. The model presented in this paper represents an extension of the simpler models of growth cycles we explored in Datta (2011) and Datta (2012).

*The paper draws extensively from the author's Ph.D. thesis, titled *Macrodynamics of Financing Investment: Applications of Lotka-Volterra Class of Models*, completed at CESP, JNU, New Delhi under the supervision of Prof. Anjan Mukherji in 2011.

We begin in section 2 by developing a baseline model of real-financial interaction, consisting of an investment function, an equation of debt dynamics and an interest rate rule. We proceed to discuss the various macroeconomic feedback effects of this model in section 3 and then discuss the local stability properties of the economically meaningful steady state in section 4. An exercise in comparative dynamics is attempted in section 5. We explore both cyclical as well as some of the other rich dynamical possibilities of this model in section 6. The main conclusions are discussed in section 7.

2 The Model

2.1 Goods Market

We consider a simple continuous time model of a closed economy, consisting of the firm and the household sector. The household sector consists of two kinds of households – type 1 households consisting of workers, deriving income from wages, and type 2 and 3 households, deriving their income from two kinds of financial assets, debt and equities respectively. The aggregate demand at time t, $AD(t)$, is composed of the total expenditure on investment and consumption made by the firms and the households respectively, i.e. $AD(t) = C(t) + I(t)$. A firm finances its investment either internally out of retained earnings, or externally by issuing debt and equity instruments. The national income, Y, might be measured by income method as the sum of wages, W, and profits, P, i.e. $Y(t) = W(t) + P(t)$. In terms of various sectors in the economy, the total income might also be represented as $Y(t) = Y_f(t) + Y_{h1}(t) + Y_{h2}(t) + Y_{h3}(t)$, where Y_f, Y_{h1}, Y_{h2} and Y_{h3} is the income to firms (profits after paying outstanding debt commitments and dividends) and type 1, type 2 and type 3 households respectively. In other words, $Y_f(t) = \sigma P(t)$, where σ is the fraction of profits retained by firms; whereas $Y_{h1}(t) = W(t)$, where W represents the wages; so that $Y_{h2}(t) = (1-\sigma) P(t) - Y_{h3}(t)$, with Y_{h2} and Y_{h3} being the part of profits representing return to financial assets (debt and equities). If s_1, s_2 and s_3 represent the fraction of the respective incomes saved by type 1, 2 & 3 households respectively, with $s_1 < s_2 = s_3$, then we have

$$C(t) = (1-s_1) W(t) + (1-s_2)(1-\sigma) P(t) \tag{1}$$

Assuming a regime of mark-up pricing, where the price per unit is obtained by adding a fixed mark-up over the wage costs of production, we have

$$P(t) = \psi Y(t) \tag{2}$$

where ψ is the share of profits in national income. Following a simple algebraic manipulation, the consumption by the household sector can now be represented as $C(t) = (1-s) Y(t)$, where $s = 1 - \{(1-s_1)(1-\psi) + (1-s_2)(1-\sigma)\psi\}$ is the propensity to save out of national income.

Let the potential output or the rate of capacity of production in the economy, $Y^\star$, be defined as the maximum output that can possibly be produced, given the existing constraints of factors and a given technology. Assuming the availability of capital as the binding constraint on production, we have $Y^\star(t) = \beta K(t)$, where β is the output-capital ratio determined by the existing technology. The actual level of output or the national income, Y, can now be represented as $Y(t) = \min(AD(t), Y^\star(t))$. In other words, for all $AD \leq Y^\star$, aggregate demand acts as the main constraint on the level of production and the output is determined by the aggregate demand.

At the goods market equilibrium, the level of output measured by the income method equals the aggregate demand, i.e. $Y(t) = AD(t)$ so that $W(t) + P(t) = C(t) + I(t)$. Substituting the value of C from (1), we have

$$Y(t) = \frac{1}{s} I(t) \tag{3}$$

Let the rate of capacity utilization be defined as the ratio of actual to potential output, i.e.

$$u(t) = \frac{Y(t)}{Y^\star(t)} \tag{4}$$

We define the rate of investment,

$$g(t) \equiv \frac{I(t)}{K(t)} \tag{5}$$

From the definition of u, $Y^\star$ and g, and the goods market equilibrium condition given in (3), we have

$$g(t) = s\beta u(t) \tag{6}$$

with a feasibility condition $0 \leq u \leq 1 \Leftrightarrow 0 \leq g \leq g_{\max}$, where $g_{\max} \equiv s\beta$ represents the rate of investment corresponding to full capacity utilization.

Let $g^\star$, the desired rate of investment, depend directly and linearly on the rate of capacity utilization, i.e. $g^\star(t) = \bar{\gamma} + \gamma(t) u(t)$. Substituting from (6), we have

$$g^\star(t) = \bar{\gamma} + \frac{\gamma(t) g(t)}{s\beta} \tag{7}$$

where γ is the 'financial accelerator' or the sensitivity of the desired rate of investment, $g^\star$ to the rate of capacity utilization, u, and is determined by financial factors. $\bar{\gamma}$, on the other hand, due to reasons given by Duménil & Lévy (1999, page 686), comprises the exogenous component of investment. Next, we turn our attention to the financial sector.

2.2 Dynamics of Debt

Consider a simple model of debt dynamics. The total stock of outstanding debt commitment in any given period, t, is given by a history of borrowing, B, at a rate of interest, $r(t)$, and repayment, R. Hence, the stock of debt in period t is given by

$$D(t) = \int_{\tau=0}^{t} (B(\tau) - R(\tau)) e^{r(t)(t-\tau)} d\tau \tag{8}$$

which, with simple algebraic manipulation and differentiation with respect to t, reduces to

$$\dot{D}(t) = B(t) - R(t) + r(t) D(t) \tag{9}$$

Equation (9) provides us with the basic accounting identity describing the growth in stock of debt. Next, we proceed to construct a macroeconomic index of financial fragility or gearing ratio, in the form of a ratio of the level of indebtedness to the ability to pay for all the debtors, i.e. the firm sector together.

In any time period, t, the firm sector's total payment commitment consists of principal and interest commitments. However, since the debt stock is accumulated over a period of time, the debtors are expected to pay only a part of the total principal in a given period. For each borrower, the minimum part of principal that is expected to be paid back in each period would differ, and would, among other things, depend on a credit rating of the borrower by the lenders. A borrower who is considered relatively safe (i.e. less likely to default) by the lenders would be expected to pay a smaller fraction of the principal in each period than a borrower who is considered relatively unsafe. In other words, borrowers with higher credit ratings will have access to loans with longer terms, resulting in a proportionally smaller minimum repayment requirements each period.

At the macroeconomic level, however, the lenders as a whole expect, in each time period, an exogenously given minimum fraction of the total debt stock as repayment towards the principal. Let this fraction be q of the total outstanding debt commitments. The interest commitments, on the other hand, are accumulated within the time period, and hence, are expected to be fully paid. In any given period t, therefore, the total minimum payment commitment of debtors is given by $(q + r) D(t)$, where $qD(t)$ and $rD(t)$ is the principal and interest component respectively. These payments are to be paid by the debtors out of their current retained profits or the internal finance. In other words, current retained profits are used to repay current payment commitments, and the residual determines the level of retained profits in the next period. The macroeconomic index of financial fragility or gearing ratio can now be represented as

$$\lambda(t) = \frac{[q + r(t)] D(t)}{\sigma P(t)} \tag{10}$$

We define

$$d(t) \equiv \frac{D(t)}{K(t)} \tag{11}$$

as the stock of debt in intensive form. Substituting from (2), (3), (5) and (11) into (10), we have

$$\lambda(t) = \frac{k[q + r(t)] sd(t)}{\sigma\psi g(t)} \tag{12}$$

The actual repayment in period t, denoted by $R(t)$, however, is independent of $[q + r(t)] D(t)$. It might either exceed or fall short of it, depending on the profile of the borrowers and repayment by individual borrowers. Let us consider a situation where a fraction $\phi(t)$ of the total outstanding debt stock is repaid in period t, i.e.

$$R(t) = \phi(t) D(t) \tag{13}$$

This fraction, $\phi(t)$ depends on:

1. The ability of the firms to repay, given by the level of retained profits, σP. A higher level of retained profits would enable the borrowers to repay a larger fraction of the outstanding debt commitments without altering it's capital structure (i.e. without taking recourse to additional external finance); and,

2. The level of the index of financial fragility, λ. Higher level of λ is associated with a borrower profile where firms, in general, have higher gearing ratios, and hence, are forced to repay back a higher fraction of outstanding debt stock. Thus, in aggregate, a higher fraction of outstanding debt stock will actually be repaid back.

Based on these considerations, we suggest the following functional form for $\phi(t)$:

$$\phi(t) = \phi(\sigma P(t), \lambda(t)) \;\; ; \quad \phi_{\sigma P} > 0, \; \phi_{\lambda} > 0 \tag{14}$$

which, taking a linear functional form, might be expressed as $\phi(t) = m\sigma P(t) \lambda(t)$, where m is constant. Substituting for the value of $P(t)$ from (2) & (3), and for the value of $\lambda(t)$ from (10), we have $\phi(t) = m(q + r)(D(t)/K(t))$, or,

$$\phi(t) = m[q + r(t)] d(t) \tag{15}$$

Next, we turn to the borrowing function, $B(t)$. In any given period t, let a fraction $a(t)$ of the total investment made by the firm sector be financed by fresh borrowing, i.e.

$$B(t) = a(t) I(t) \tag{16}$$

The fraction, $a(t)$, will be determined by the financial structure of the firms, i.e. the manner in which the firms decide to finance fresh investments. To arrive at a particular level of $a(t)$ the firms need to take two kinds of decisions: (a) the decision on distribution of the cost of investment between internal (i.e. retained profits) and external (i.e. debt and outside equities) sources of finance; and, (b) the decision on how to distribute the proportion of investment costs marked for external source between debt and equity financing. We first note the following:

Proposition 1. *For a given level of profits, a higher rate of investment would necessarily mean a higher level of outside sources of finance.*

Proof. Following a flow of funds approach, we note that the firm sector receives its funds from retained profits, borrowing and equity financing, and uses these funds in making planned investment, paying out outstanding debt commitments, and in unplanned accumulation of inventories, i.e. $\sigma P(t) + B(t) + E(t) \equiv I(t) + R(t) + \Delta N(t)$, where $\Delta N(t)$ represents the unplanned accumulation of inventories by the firm sector in period t. Substituting from (2), (3), (13) and (15), we have

$$B(t) + E(t) \equiv \left(1 - \frac{\psi}{s}\right) I(t) + m\left[q + r(t)\right]\{d(t)\}^2 K(t) + \Delta N \tag{17}$$

$$\Rightarrow \frac{\partial\left(B(t) + E(t)\right)}{\partial I(t)} \equiv \left(1 - \frac{\psi}{s}\right) > 0 \tag{18}$$

In other words, for a given level of profits, higher the level of investment higher would be the use of outside sources of finance like debt and outside equities. □

Further, though a detailed analysis of equity financing is beyond the scope of our analysis, we note the following:

Remark 1. *Between two sources of external finance, there might be an increasing preference for debt as the rate of investment increases.*

Remark 1 could be explained by the following:

1. The main difference between debt and equities is with regard to the resulting payment commitments. While the payment commitments arising out of debt commitments, consisting of the principal and the interest, is independent of profits, the payment commitments arising out of equity financing, consisting of dividends, $(1 - \sigma) P(t)$, depend directly on profits. Hence, in periods of prosperity, characterized by a high rate of both investment and profits (related through the multiplier from (3)), cost of equity financing would be higher. In other words, any increase in investment would increase the cost of equity financing faster than the cost of debt financing.

2. Further, as increases in investment leads to increased recourse to external financing from proposition 1, the managers of the firms might be averse to continue increasing the dilution of shareholding from equity financing. Since a dilution of shareholding, by changing the ownership structure, increases the threat of hostile takeovers and change in corporate controls (provided, of course, such markets exist), managers might prefer debt financing when the requirement of external financing is higher.

It should be pointed out that proposition 1 and remark 1, taken together, establishes a direct relationship between the fraction of investment cost in any period, $a(t)$, financed by debt, $B(t)$. In addition, we also note the following:

Remark 2. *An increase in the level of financial fragility, λ, might necessitate financing a higher proportion of the cost of investment through debt.*

We should note that remark 2 is motivated by the relationship implied in (14). A higher level of financial fragility, λ, from (14), will imply that a higher fraction outstanding debt commitments will have to be repaid in the current period. This will require a higher level of borrowing, to be used not only towards meeting the cost of investment but also towards repaying outstanding debt commitments.

From proposition 1 and remark 1 and 2, we suggest the following functional form for $a(t)$:

$$a(t) = a(g(t), \lambda(t)) \ ; \quad a_g > 0, \ a_\lambda > 0 \tag{19}$$

which, taking a linear functional form and substituting from (2), (3) and (12), might be expressed as

$$a(t) = \frac{k[q + r(t)]\, s}{\sigma\psi} d(t) \tag{20}$$

Substituting from (13), (15), (16) and (20) into (9), we have

$$\dot{d}(t) = \left[\left\{\frac{k(q + r(t))\, s}{\sigma\psi} - 1\right\} g(t) - m(q + r(t))\, d(t) + r(t)\right] d(t) \tag{21}$$

2.3 Financial Determinants of Investment

We now turn our attention to the financial determinants of the rate of investment. To begin with, we suggest that the financial accelerator, γ, depends inversely on the rate of interest. This could be because

(a) an increase in the rate of interest increases the the cost of servicing debt for firms financing a part of their investment through debt;

(b) an increase in the rate of interest increases the opportunity cost of investing in physical capital; and

(c) an increase in the rate of interest, by increasing the likelihood of an adverse selection of risky projects might lead to an increase in credit rationing and red-lining and abandonment of projects which might have been feasible at a lower rate of interest.

Next, we turn to other financial determinants of investment.

Consider the process of assessment of loan application by lenders. Any decision on such an application, in the form of an approval or lack of it, would involve a detailed analysis of the creditworthiness of the loan application. While the actual process of an assessment of creditworthiness can be quite complicated[1], we consider a simple version of this process here. Broadly, the quantitative factors determining the creditworthiness of a loan application might be categorized into two classes: those which remain unchanged across various stages of a business cycle; and, those which vary as an economy moves through a business cycle. In the first category, which might be considered as a preliminary assessment by the lending

[1] See, for instance, Kalapodas & Thomson (2006) and Abrahams & Zhang (2009) for a discussion of the process of credit risk assessment.

institutions, we might include permanent factors like the credit history and reputation of an individual, or a group of individuals. Based on these factors, the lending institutions assign a credit rating or score to the borrowers. A borrower might be classified as either *prime* or *sub-prime* through such a process. Once classified, the identity of a borrower does not change across various stages of the business cycle; in other words, a change in the rate of capacity utilization will have no impact on this identity of the borrower. However, the final decision on creditworthiness, in addition to above, is also likely to consider an additional component that includes current determinants. This would include, for instance, the current income of the loan applicant and an assessment of the expected future income. Assessment of future income might include, among other things, the expected profitability and risk associated with the investment project for which the borrower seeks a loan. As would be evident, these factors would vary across various stages in a business cycle; in particular, it would depend on the current rate of capacity utilization.

We begin by attempting to formalize the first, i.e. the fixed component of creditworthiness. As we noted above, this depends on an individual credit rating of each borrower. Consequently, consider the portfolio of a lender; this portfolio will be characterized by a certain spread of prime or safe, and sub-prime or risky borrowers. This might be formalized by introducing η, an indicator of the proportion of borrowers with high perceived risk of default in the overall debt portfolio, such that $\eta \in [0, 1]$. A higher value of η would imply a greater proportion of borrowers with high perceived risk of default in the macroeconomic distribution of debt.

It is often argued[2] that periods of relative prosperity might be accompanied with a gradual worsening of the profile of borrowers, leading to inclusion of borrowers with higher perceived risk of default (i.e. the sub-prime borrowers). This inclusion of sub-prime borrowers would be quite evident if the prudential norms followed by the lenders are fixed at an absolute level. For instance, if having access to a particular value of loan requires furnishing a fixed amount of collateral, it is clear that a greater number of potential borrowers would be able to provide the required collaterals, and hence, have access to loan in periods of prosperity. In other words, those excluded by the debt market during periods with lower levels of economic activity would be included during periods of prosperity. The prudential norms, however, typically do not remain fixed but, in fact, are relaxed during periods of prosperity, because of optimistic expectations. Apart from a direct relaxation, financial innovation and predatory lending practices by organized lenders during a boom and emergence of new financial instruments might aid such relaxation of prudential norms during periods of prosperity[3]. This reinforces the impact of a phase of prosperity in increasing the proportion of risky borrowers in the macroeconomic distribution of debt.

[2] See, for instance, Fisher (1932, 1933), Minsky (1975, 1982, 1986, 1994).

[3] See, for instance, Kregel (2008), Shiller (2008), Abrahams & Zhang (2009), Reinhart & Rogoff (2009), Akerlof & Shiller (2010).

Next, we formalize the above argument. Since the period of prosperity, as defined throughout our analysis, is characterized by an increase in u, Y and g, we suggest the following functional formulation for the proportion of risky borrowers, η in the portfolio:

$$\eta(t) = \eta_g g(t) \tag{22}$$

where η_g is a constant such that $\eta_g \in \left]0, \frac{1}{g_{\max}}\right]$.

We now construct a cumulative index of risk of default by including the impact of η, as defined above in (22), and the macroeconomic indicator of financial fragility, λ, as defined in (10), as follows:

$$\Lambda(t) = \Lambda_\eta \eta(t) + \Lambda_\lambda \lambda(t) \tag{23}$$

where Λ_η and Λ_λ represent the sensitivity of the cumulative index of risk of default to η and λ respectively.

One should note that the cumulative index of risk of default, Λ, consist of two separate risk components. These two components might be interpreted as emerging from two different kinds of risks involved in credit expansion. The first, or the proportion of risky borrowers in the macroeconomic distribution of debt or η, might be considered an indicator of risk involved in credit widening, i.e. inclusion of new borrowers, some of whom might be considered subprime. The second, the macroeconomic indicator of financial fragility or λ, on the other hand, might be considered a more conventional financial ratio that takes into account both credit deepening and credit widening. Hence, taken together, Λ might be considered a more comprehensive macroeconomic indicator of risk of default than some of the conventional indicators, since it takes into account both credit deepening and credit widening.

There are two ways the rate of investment might be affected by the risk of default. Firstly, as we have argued before, the managers are concerned with the risk of default, since in case of a default, a firm might face a hostile takeover, leading to a change in corporate control threatening the job of the managers. Hence, an increase in Λ might prompt the managers to respond by reducing the sensitivity of the rate of investment to the capacity utilization, i.e. the accelerator. Secondly, the lenders are concerned with the risk of default. An increase in a macroeconomic indicator of the risk of default like Λ is likely to make them more cautious about lending. In light of a substantial literature in this area (see, for instance Kalecki 1937, Hodgman 1960, Catt 1965, Stiglitz & Weiss 1981, Stiglitz & Weiss 1983, Jaffee & Stiglitz 1990, Stiglitz & Weiss 1992), we might note that a rationing and red-lining of credit might be one of the possible responses from the lenders under such a situation. While such a rationing and red- lining will directly affect only a section of borrowers, all borrowers are likely to take steps to reduce the possibility of being rationed and red-lined. Since individual or firm-level gearing ratio is one of the deciding factors on which firms are rationed or red-lined, an increase in Λ is likely to induce individual firms to to respond by trying to reduce their gearing ratios. Since this logic applies to all the firms, an increase in Λ will have a negative impact on the accelerator of the investment function.

We formalize the arguments given in the paragraphs by introducing the following formulation for the accelerator:

$$\gamma(t) = \bar{\mu} - \hat{\mu}\Lambda(t) - \alpha r(t) \tag{24}$$

where $\hat{\mu}$ is the sensitivity of the accelerator to the cumulative risk of default, α is the sensitivity of the accelerator to the rate of interest, and $\bar{\mu}$ represents the maximum possible level of the accelerator. Substituting the values of $\lambda(t)$ and $\eta(t)$ from (12) and (22) into (23), and then substituting the resultant expression into (24), we have

$$\gamma(t) = \bar{\mu} - \hat{\mu}\Lambda_\eta \eta_g g(t) - \frac{\hat{\mu}\Lambda_\lambda k\{q + r(t)\}\, s}{\sigma\psi}\frac{d(t)}{g(t)} - \alpha r(t) \tag{25}$$

Substituting the value of accelerator, $\gamma(t)$, from (25) into the investment function in (7), we have

$$g^\star(t) = \bar{\gamma} + \frac{1}{s\beta}\left[\bar{\mu} - \hat{\mu}\Lambda_\eta \eta_g g(t) - \frac{\hat{\mu}\Lambda_\lambda k\{q + r(t)\}\, s}{\sigma\psi}\frac{d(t)}{g(t)} - \alpha r(t)\right] g(t) \tag{26}$$

Let the rate of investment be continuously adjusted so as to meet a fraction, h, of the gap between the actual and the desired rate of investment, i.e.

$$\frac{\dot{g}(t)}{g(t)} = h\left(g^\star(t) - g(t)\right) \tag{27}$$

where h represents the speed of adjustment of the actual investment to the desired level by the investors. Substituting the value of $g^\star(t)$ from (26) into (27), we have the following equation of motion to represent the dynamics of the rate of investment:

$$\begin{aligned}\dot{g}(t) = \Bigg[\left(\frac{\bar{\mu}}{s\beta} - 1\right) g(t) - \frac{\hat{\mu}\Lambda_\eta\eta_g}{s\beta}\{g(t)\}^2 - \frac{\hat{\mu}\Lambda_\lambda kq}{\sigma\psi\beta} d(t) - \frac{\hat{\mu}\Lambda_\lambda k}{\sigma\psi\beta} r(t)\, d(t) \\ - \frac{\alpha}{s\beta} g(t)\, r(t) + \frac{\bar{\gamma}}{s\beta}\Bigg] h g(t)\end{aligned} \tag{28}$$

2.4 Interest Rate Rules

The rate of interests in our model is determined by the Central Bank, following a modified version of Taylor Rule that targets the level of capacity utilization and adjusts the rate of interest as a response to the gap between the target and the actual level of capacity utilization. If the level of capacity utilization desired by the Central Bank is represented by $u^\star$ (where $0 < u^\star \leq 1$), then its interest rate rule is given by

$$\begin{aligned}\dot{r}(t) &= l\left(u - u^\star\right) \\ \Rightarrow \dot{r}(t) &= l\left\{\frac{g(t)}{s\beta} - u^\star\right\} r(t)\end{aligned} \tag{29}$$

where l is the speed of adjustment of the rate of interest by the Central Bank.

2.5 Final Model

Equation (21), (28) and (29) together completely describes our dynamical system, reproduced below for clarity:

$$\boxed{\begin{aligned}
\dot{g}(t) &= \left[\left(\frac{\bar{\mu}}{s\beta}-1\right)g(t)-\frac{\hat{\mu}\Lambda_\eta\eta_g}{s\beta}\{g(t)\}^2-\frac{\hat{\mu}\Lambda_\lambda kq}{\sigma\psi\beta}d(t)-\frac{\hat{\mu}\Lambda_\lambda k}{\sigma\psi\beta}r(t)\,d(t)\right.\\
&\qquad\left.-\frac{\alpha}{s\beta}g(t)\,r(t)+\frac{\bar{\gamma}}{s\beta}\right]hg(t)\\
\dot{r}(t) &= l\left\{\frac{g(t)}{s\beta}-u^\star\right\}r(t)\\
\dot{d}(t) &= \left[\left(\frac{kqs}{\sigma\psi}-1\right)g(t)+\frac{ks}{\sigma\psi}g(t)\,r(t)-mqd(t)-mr(t)\,d(t)+r(t)\right]d(t)
\end{aligned}} \tag{30}$$

It might be noted that dynamics underlying the system represented by (30) has similarities with ecological models in the class of generalized three-dimensional Lotka-Volterra system with two predators and one prey. This class of models have been widely studied in the literature in the field of mathematical ecology.[4] The debt-capital ratio, d, and the rate of interests, r might be seen to be analogous to the population of predators, whereas the rate of investment, g might be seen to be analogous to the population of prey. Underlying such a analogy with models in ecology, however, there is a complex interaction of several macroeconomic feedback effects. Before we proceed with our discussion of the solution to the dynamical system represented by (30), we provide a brief discussion of these macroeconomic feedback effects.

3 Macroeconomic Feedback Effects

Consider the dynamical system represented by (30) in steady state, when it is disturbed by an increase in the rate of investment, g. The total effect as a result of this increase in the rate of investment might be decomposed into several macroeconomic feedback effects. We provide a list of few of these effects below, by isolating each of them and ignoring all the other effects.

1. Multiplier-Accelerator Relationship: There is a *positive* feedback effect resulting from an interaction between the multiplier (a positive feedback from investment to output) and the accelerator (a positive feedback from output to investment via demand). The actual working of this effect is shown below:

$$g\uparrow\xrightarrow{\text{multiplier}}Y\uparrow\xrightarrow{\text{by (4)}}u\uparrow\xrightarrow{\text{accelerator}}g^\star\uparrow\xrightarrow{\text{by (27)}}g\uparrow$$

[4] See, for instance, Koch (1974), Butler & Waltman (1981), Smith (1982), Cushing (1984), Gardini, Lupini & Messia (1989), Hofbauer & So (1994), Korobeinikov & Wake (1999), Hsu, Hwang & Kuang (2001), Loladze, Kuang, Elser & Fagan (2004), Feng & Hinson (2005); a collection of results for a class of such systems might be found in Zeeman (1993) and Zeeman & Zeeman (2002); for a more general discussion in the context of dynamical possibilities in three-dimensional differential equation systems, see Kuznetsov (1997).

2. Financial feedback I: There is a *negative* feedback effect arising from the interaction of the investment function with the Taylor-type interest rate rule. Due to reasons discussed above, the rate of interest has an inverse effect on the rate of investment. Rate of investment, on the other hand, has a positive effect on the rate of interest, since any increase in the rate of investment, *ceteris paribus* leads to an increase in the rate of capacity utilization, making the monetary authorities upwardly revise the rate of interest (provided, of course, one started out from a situation of equilibrium where the rate of capacity utilization was equal to the rate desired by the monetary authorities.). The working of this feedback effect is shown below:

$$g \uparrow \xrightarrow{\text{multiplier}} Y \uparrow \xrightarrow{\text{by (4)}} u \uparrow \xrightarrow{\text{Taylor rule}} r \uparrow \xrightarrow{\text{investment function}} g^{\star} \downarrow \xrightarrow{\text{by (27)}} g \downarrow$$

3. Financial feedback II: There is a *negative* feedback effect arising from the a rise in the rate of investment worsening the profile of borrowers in the macroeconomic distribution of debt. As noted above, an increase in the rate of investment leads to an increase in the proportion of risky or subprime borrowers in the macroeconomic distribution of debt. This increases the cumulative index of risk of default, which in turn has a negative impact on the rate of investment through the investment function. The working of this feedback effect is shown below:

$$g \uparrow \xrightarrow{\text{by (22)}} \eta \uparrow \xrightarrow{\text{by (23)}} \Lambda \uparrow \xrightarrow{\text{investment function}} g^{\star} \downarrow \xrightarrow{\text{by (27)}} g \downarrow$$

4. Financial feedback III: There is a *negative* feedback effect arising from a rise in investment inducing a greater fraction of investment being financed by borrowing, leading to a rise in debt-capital ratio and the macroeconomic indicator of financial fragility, eventually dampening the rate of investment through the investment function. The working of this feedback effect is shown below:

$$g \uparrow \longrightarrow B \uparrow \longrightarrow d \uparrow \xrightarrow{\text{by (12)}} \lambda \uparrow \xrightarrow{\text{by (23)}} \Lambda \uparrow \xrightarrow{\text{investment function}} g^{\star} \downarrow \xrightarrow{\text{by (27)}} g \downarrow$$

5. Secondary financial feedback: There are at least two secondary *negative* feedback effects arising from the interaction of the investment function with the Taylor-type interest rate rule. As mentioned above, an increase in the rate of investment would induce the monetary authorities to increase the rate of interest. From thereon, however, in addition to the direct feedback effect on the rate of investment discussed above, there are two additional feedback effects. Firstly, an increase in the rate of interest increases the debt repayment commitments, leading to a rise in the macroeconomic indicator of financial fragility. This will dampen the rate of investment through the investment function. Secondly, an increase in the rate of investment also increases the borrowing requirements. This will increase the debt-capital ratio, and hence, the macroeconomic indicator of financial fragility, once again dampening the rate of investment. These two

feedback effects are shown below:

$$
\begin{aligned}
(a) \quad & g\uparrow \xrightarrow{\text{multiplier}} Y\uparrow \xrightarrow{\text{by (4)}} u\uparrow \xrightarrow{\text{Taylor rule}} r\uparrow \xrightarrow{\text{by (12)}} \lambda\uparrow \xrightarrow{\text{by (23)}} \Lambda\uparrow \\
& \xrightarrow{\text{investment function}} g^\star\downarrow \xrightarrow{\text{by (27)}} g\downarrow \\
(b) \quad & g\uparrow \xrightarrow{\text{multiplier}} Y\uparrow \xrightarrow{\text{by (4)}} u\uparrow \xrightarrow{\text{Taylor rule}} r\uparrow \longrightarrow B\uparrow \longrightarrow d\uparrow \xrightarrow{\text{by (12)}} \lambda\uparrow \\
& \xrightarrow{\text{by (23)}} \Lambda\uparrow \xrightarrow{\text{investment function}} g^\star\downarrow \xrightarrow{\text{by (27)}} g\downarrow
\end{aligned}
$$

4 Steady State and Local Stability

We begin by first noting that the dynamical system represented by (30) has only one economically meaningful, i.e. interior non-trivial steady state, $E_7 : \left(\bar{g}_7, \bar{r}_7, \bar{d}_7\right)$, such that $E_7 \in \textbf{int}\left(\Re^3_{+++}\right)$,[5] with six other steady states having a trivial steady state value for at least one of the variables. In the following sections, we confine our analysis only to the local dynamics in the neighborhood of the economically meaningful steady state.

Using the computer algebra system, Maxima[6], we determine the non-trivial steady state of the dynamical system represented by (30) to be the following:

$$
\begin{aligned}
E_7 &: \left(\bar{g}_7, \bar{r}_7, \bar{d}_7\right) \\
&= \left(s\beta u^\star,\ \frac{\hat{\mu}\eta_g\Lambda_\eta s^2\beta^2\sigma^2\psi^2 m(u^\star)^2+\left\{\Lambda_\lambda\hat{\mu}k^2qs^3\beta+\left(\beta^2\sigma^2\psi^2m-\hat{\mu}\Lambda_\lambda k\beta\sigma\psi\right)s^2-\bar{\mu}s\beta\sigma^2\psi^2m\right\}u^\star-\bar{\gamma}\sigma^2\psi^2m}{(\hat{\mu}\Lambda_\lambda k^2s^3\beta+\alpha s\beta\sigma^2\psi^2m)u^\star+\hat{\mu}\Lambda_\lambda ks\sigma\psi}, \right. \\
&\left. \frac{\hat{\mu}\eta_g\Lambda_\eta ks^4\beta^3(u^\star)^3+\left\{ks^4\beta^3-(\alpha q+\bar{\mu})ks^3\beta^2+(\alpha+\hat{\mu}\eta_g\Lambda_\eta)s^2\beta^2\sigma\psi\right\}(u^\star)^2+\left\{(\beta\sigma\psi-\bar{\gamma}k)s^2\beta-\bar{\mu}\beta\sigma\psi s\right\}u^\star-\bar{\gamma}\sigma\psi}{\hat{\mu}\eta_g\Lambda_\eta s^2\beta^2\sigma\psi m(u^\star)^2+\{(\beta\sigma\psi m-\hat{\mu}\Lambda_\lambda k)s^2\beta-(\alpha q+\bar{\mu})s\beta\sigma\psi m\}u^\star-\hat{\mu}\Lambda_\lambda kqs-\bar{\gamma}\sigma\psi m} \right)
\end{aligned}
\tag{31}
$$

i.e.

$$
\begin{aligned}
\bar{g}_7 &= s\beta u^\star \\
\bar{r}_7 &= \frac{\hat{\mu}\eta_g\Lambda_\eta s^2\beta^2\sigma^2\psi^2 m(u^\star)^2+\left\{\Lambda_\lambda\hat{\mu}k^2qs^3\beta+\left(\beta^2\sigma^2\psi^2m-\hat{\mu}\Lambda_\lambda k\beta\sigma\psi\right)s^2-\bar{\mu}s\beta\sigma^2\psi^2m\right\}u^\star-\bar{\gamma}\sigma^2\psi^2m}{(\hat{\mu}\Lambda_\lambda k^2s^3\beta+\alpha s\beta\sigma^2\psi^2m)u^\star+\hat{\mu}\Lambda_\lambda ks\sigma\psi} \\
\bar{d}_7 &= \frac{\hat{\mu}\eta_g\Lambda_\eta ks^4\beta^3(u^\star)^3+\left\{ks^4\beta^3-(\alpha q+\bar{\mu})ks^3\beta^2+(\alpha+\hat{\mu}\eta_g\Lambda_\eta)s^2\beta^2\sigma\psi\right\}(u^\star)^2+\left\{(\beta\sigma\psi-\bar{\gamma}k)s^2\beta-\bar{\mu}\beta\sigma\psi s\right\}u^\star-\bar{\gamma}\sigma\psi}{\hat{\mu}\eta_g\Lambda_\eta s^2\beta^2\sigma\psi m(u^\star)^2+\{(\beta\sigma\psi m-\hat{\mu}\Lambda_\lambda k)s^2\beta-(\alpha q+\bar{\mu})s\beta\sigma\psi m\}u^\star-\hat{\mu}\Lambda_\lambda kqs-\bar{\gamma}\sigma\psi m}
\end{aligned}
$$

We should note that the economically meaningful steady state value of g, $\bar{g}_7$, is completely determined only from the equation of motion for r in (30). In other words, the interest rate rule followed by the Central Bank determines the steady state rate of capacity utilization at $u^\star$, and consequently, the steady state rate of investment at $s\beta u^\star$. We further note that $\bar{g}_7$ satisfies the economic feasibility condition, since $u^\star \in\ ]0, 1[\ \Leftrightarrow \bar{g}_7 = s\beta u^\star < s\beta = g_{\max}$. We

[5]This would be obvious by calculating the non-trivial steady state value of g from the equation of motion of r in (30), and substituting this steady state value of g in the equation of motion of g and d to calculate the non-trivial steady state values for r and d.

[6]Maxima version 5.20.1, using Lisp SBCL 1.0.29.11.debian, distributed under the GNU Public License (http://maxima.sourceforge.net).

next turn our attention to the local stability properties of the economically meaningful steady state, E_7.

Linearizing (30) around the economically meaningful steady state, E_7, from the first-order term of its Taylor expansion evaluated at E_7, we have

$$\begin{bmatrix} \dot{g}(t) \\ \dot{r}(t) \\ \dot{d}(t) \end{bmatrix} \approx J_{E_7} \begin{bmatrix} g(t) - \bar{g}_7 \\ r(t) - \bar{r}_7 \\ d(t) - \bar{d}_7 \end{bmatrix} \tag{32}$$

where

$$J_{E_7} = \begin{bmatrix} (\bar{\mu} - s\beta - 2\hat{\mu}\Lambda_\eta \eta_g \bar{g}_7 - \alpha \bar{r}_7)\, hu^\star & -\left(\frac{\hat{\mu}\Lambda_\lambda k}{\sigma\psi\beta}\bar{d}_7 + \alpha u^\star\right) h\bar{g}_7 & -\frac{\hat{\mu}\Lambda_\lambda k\,(q+r)}{\sigma\psi\beta} h\bar{g}_7 \\ \frac{l}{s\beta}\bar{r}_7 & 0 & 0 \\ \left\{\frac{ks}{\sigma\psi}(q+\bar{r}_7) - 1\right\}\bar{d}_7 & \left(\frac{ks}{\sigma\psi}\bar{g}_7 - m\bar{d}_7 + 1\right)\bar{d}_7 & -m\,(q+\bar{r}_7)\,\bar{d}_7 \end{bmatrix} \tag{33}$$

is the jacobian matrix evaluated at the steady state, E_7.

Next, we calculate the characteristic equation to the jacobian matrix given in (33) and test for the conditions for local stability from Routh-Hurwitz condition. Following (Flaschel 2009, page 385, theorem A.5) for the characteristic equation represented by $\vartheta^3 + a_1\vartheta^2 + a_2\vartheta + a_3 = 0$, all the eigenvalues have negative real parts if and only if the set of inequalities $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$ is satisfied. We test for these inequalities and obtain the following:

1. $a_1 > 0 \quad \forall \quad h > \dfrac{m\,(q+\bar{r}_7)\,\bar{d}_7}{(\bar{\mu} - s\beta - \alpha\bar{r}_7 - 2\hat{\mu}\Lambda_\eta\eta_g s\beta u^\star)\,u^\star}$
2. $a_3 > 0$ as long as all the parameters, including those representing the rates of adjustment, i.e. h and l, are positive.
3. The expression, $a_1a_2 - a_3$ is a polynomial, which is linear in l and quadratic in h. Typically, one can fix any one of the parameters, say h and calculate a critical range of the monetary policy parameter, l, within which this expression will take a positive value.

Thus, one can expect that possibilities of local convergence to the economically meaningful steady state, E_7, exists for a reasonable set of parameter values. For instance, if the parameters have values as follows:

$$\begin{array}{lllllll} s = 0.3, & \sigma = 0.4, & \psi = 0.3, & \alpha = 0.5, & q = 0.6, & m = 0.6, & k = 0.7, \\ \beta = 0.8, & \bar{\mu} = 0.3, & \hat{\mu} = 0.4, & \eta_g = 0.1, & \Lambda_\eta = 0.1, & \Lambda_\lambda = 0.9, & \bar{\gamma} = 0.5 \end{array}$$

and if the monetary authorities are targeting a capacity utilization of 80%, i.e. $u^\star = 0.8$, and if the rate of adjustment of investment by private sector is fixed at any positive value of h, say, at $h = 0.8$, then it turns out that $a_1,\ a_3 > 0$. Further, $a_1 a_2 - a_3 > 0$ if the monetary policy parameter is kept below a critical value, $l < 0.367$. For instance, $l = 0.3$ along with $h = 0.8$ will satisfy all the three conditions listed above, and the economically meaningful steady state, $E_7\left(\bar{g}_7, \bar{r}_7, \bar{d}_7\right)$, which in this case will attain a value $\approx (0.192, 0.334, 0.814)$ will be locally asymptotically stable.

5 Convergence and Comparative Dynamics

In this section, we attempt a comparative dynamic analysis to determine the sensitivity of the non-trivial steady state, E_7 with respect to some of the parameters. For this purpose, we restrict our analysis to the case where the non-trivial steady state, E_7 is locally asymptotically stable. We note, first of all, that the economically meaningful long-run steady state is completely insensitive to the rate of adjustment parameters, viz. h and l, representing the rate of adjustment of investment by the private sector and that of the rate of interest by the central bank respectively. This should not come across as surprising. The process of adjustment, both by the private investors and the central bank takes place in the short-run. Since long-run steady state is attained only after the process of adjustment has taken place, the rates of adjustment by both private sector and the central bank has no impact on the long-run steady state, as expected.

5.1 Factors affecting steady state rate of growth

We recall from (31) that at the non-trivial steady state, the rate of investment, $\bar{g}_7 = s\beta u^\star$. It follows that

$$\frac{\partial \bar{g}_7}{\partial s} = \beta u^\star > 0 \tag{34a}$$

$$\frac{\partial \bar{g}_7}{\partial \beta} = s u^\star > 0 \tag{34b}$$

$$\frac{\partial \bar{g}_7}{\partial u^\star} = s\beta > 0 \tag{34c}$$

i.e. $\bar{g}_7$ depends directly on the propensity to save, s, the output-capital ratio determined by the technology, β, and the level of capacity utilization desired by the Central Bank, $u^\star$. We further note that $\partial \bar{g}_7/\partial \psi > 0$ and $\partial \bar{g}_7/\partial \sigma > 0$, i.e. $\bar{g}_7$ depends directly on the share of profits in national income, ψ, and the fraction of profits retained by the firm sector, σ.

The above results would need some clarifications in the context of a long-standing debate in macroeconomics between the post-Keynesians on one side and both mainstream Keynesians as well as Harrodians or neo-Marxians on the other. In both the mainstream Keynesian as well as the Harrodian or neo-Marxian literature, an exogenously given rate of growth of output or rate of capacity utilization is explicitly included in the model. Such an exogenously given rate of growth of output or rate of capacity utilization is known variously as the ‘normal rate’, the ‘natural rate’ or ‘non-accelerating inflation rate’. This exogenously given rate of growth is typically included either in the investment function or in the specification of phillips curve in such a way that in the long-run steady state, the economy grows along this exogenously specified rate. Post-Keynesians, on the other hand, argue that such a 'normal' or 'natural' rate either does not exist, or even if it exists, it must be determined endogenously from the short-run growth rates. Empirical studies by Leon-Ledesma & Thirlwall (2000, 2002, 2007) also seem to support such a position.

In light of the above debate, the comparative dynamics results for our model might seem rather unexpected. We started out with a fairly standard post-Keynesian relationship between the capacity utilization and the rate of investment, which was then augmented by financial factors. Yet, the steady state rate growth looks similar to those in Harrodian class of models. For instance, in Harrod's (1939) original model, the economy grows in steady state at the rate of $s\beta$. The steady state rate of growth in our formulation, $s\beta u^\star$, like the Harrodian models, is directly proportional to the propensity to save out of national income and the technological output-capital ratio.[7] In other words, the paradox of thrift operates only in the short-run but not in the long-run. In this sense, our results are closer to both the mainstream Keynesian and Harrodian literature, as opposed to the standard post-Keynesian result that paradox of thrift operates both in the short-run as well as the long-run.

This apparent similarity that the long-run results of our model exhibits with the New-Keynesian and Harrodian models, however, needs some qualifications. On a closer examination, our model shows an important point of departure from these classes of models. We note that $u^\star$ is policy determined, and represents the target rate of capacity utilization by the central bank. In this sense, the rate of capacity utilization at the steady state in our model is neither at the full employment level, as in Harrod (1939), nor at the exogenously given 'natural' or 'normal' rate to which the private investors try to adjust their actual rate of investment. A corollary from this would follow, that the monetary authorities are in a position to influence the long-run steady state rate of growth, unlike the New-Keynesian or the Harrodian models, although such an ability to influence the steady state will be limited by restrictions imposed from the stability conditions discussed above.

5.2 Factors affecting steady state rate of interest

We recall from (31) that at the non-trivial steady state, the rate of interest,

$$\bar{r}_7 = \frac{\hat{\mu}\eta_g\Lambda_\eta s^2\beta^2\sigma^2\psi^2 m(u^\star)^2 + \left\{\Lambda_\lambda\hat{\mu}k^2 q s^3\beta + \left(\beta^2\sigma^2\psi^2 m - \hat{\mu}\Lambda_\lambda k\beta\sigma\psi\right)s^2 - \bar{\mu}s\beta\sigma^2\psi^2 m\right\}u^\star - \bar{\gamma}\sigma^2\psi^2 m}{(\hat{\mu}\Lambda_\lambda k^2 s^3\beta + \alpha s\beta\sigma^2\psi^2 m)u^\star + \hat{\mu}\Lambda_\lambda k s\sigma\psi}$$

We first consider the effect of a small change in the parameter $u^\star$, representing the rate of capacity utilization targeted by the central bank, on the steady state rate of interest, $\bar{r}_7$.

[7] In fact, if the monetary authorities in our model target full capacity utilization, then the rate of investment in the steady state of our model would algebraically coincide with that of Harrod's (1939) model.

Partially differentiating $\bar{r}_7$ with respect to $u^\star$, we have

$$\begin{aligned}\frac{\partial \bar{r}_7}{\partial u^\star} = &- \Big[\beta\sigma\psi \Big\{\hat{\mu}^2\eta_g\Lambda_\eta\Lambda_\lambda\beta^2\sigma\psi k^2 m s^4 u^{\star 2} + \hat{\mu}\eta_g\Lambda_\eta\alpha\beta^2\sigma^3\psi^3 m^2 s^2 u^{\star 2} + \bar{\gamma}\alpha\sigma^3\psi^3 m^2 \\ &+ \left(\bar{\gamma}ks - \sigma\psi\left(1 - s\beta\right)\right)\hat{\mu}\Lambda_\lambda\sigma\psi kms + \left(kqs - \sigma\psi\right)\hat{\mu}{\Lambda_\lambda}^2 k^2 s^2 \\ &+ 2\hat{\mu}^2\eta_g\Lambda_\eta\Lambda_\Lambda\beta\sigma^2\psi^2 ks^2 mu^\star\Big\}\Big] \Big/ \Big[s\left(\hat{\mu}\Lambda_\lambda\beta k^2 s^2 u^\star + \alpha\beta\sigma^2\psi^2 mu^\star + \hat{\mu}\Lambda_\lambda k\sigma\psi\right)^2\Big]\end{aligned} \tag{35}$$

We recall that $kqs - \sigma\psi > 0$, hence, it follows that $\dfrac{\partial \bar{r}_7}{\partial u^\star} < 0$ provided

$$\bar{\gamma} > \frac{\sigma\psi}{ks}\left(1 - s\beta\right) \tag{36}$$

where both $\sigma\psi/ks$ and $1 - s\beta$ are less than 1. We note that (36) provides a sufficient (though not necessary) condition for the steady state rate of interest, $\bar{r}_7$ to have an inverse relation with the level of capacity utilization targeted by the Central Bank, $u^\star$. In other words, if (36) holds true,[8] then targeting a higher rate of capacity utilization by the monetary authorities would imply that the rate of interest will be lower at the steady state.

Next, we consider the effect of a small change in the parameter $\bar{\mu}$ on the steady state rate of interest, $\bar{r}_7$. Partially differentiating $\bar{r}_7$ with respect to $\bar{\mu}$, we have

$$\frac{\partial \bar{r}_7}{\partial \bar{\mu}} = \frac{\beta\sigma^2\psi^2 mu^\star}{\left(\hat{\mu}\Lambda_\lambda\beta k^2 s^2 + \alpha\beta\sigma^2\psi^2 m\right)u^\star + \hat{\mu}\Lambda_\lambda\sigma\psi k} > 0 \tag{37}$$

The parameter, $\bar{\mu}$ refers to the non-financial component of the investment accelerator, i.e. which is affected only by capacity utilization and is not dampened by any of the financial factors. As we noted earlier, financial dampeners play a role in stabilizing the the relationship between the multiplier and the accelerator. Since $\bar{\mu}$ represents the part of the accelerator unaffected by the stabilizing impact of financial dampeners, a higher level of $\bar{\mu}$ would have destabilizing implications and would require more contractionary monetary policy in the form of a higher rate of interest in the steady state in order to bring the economy to its target rate of capacity utilization.

Next, we consider the effect of a small change in the parameter η_g on the steady state rate of interest, $\bar{r}_7$. Partially differentiating $\bar{r}_7$ with respect to η_g, we have

$$\frac{\partial \bar{r}_7}{\partial \eta_g} = -\frac{\hat{\mu}\Lambda_\eta\beta^2\sigma^2\psi^2 msu^{\star 2}}{\left(\hat{\mu}\Lambda_\lambda\beta k^2 s^2 + \alpha\beta\sigma^2\psi^2 m\right)u^\star + \hat{\mu}\Lambda_\lambda\sigma\psi k} < 0 \tag{38}$$

We recall that η_g refers to the rate at which the proportion of risky borrowers in the macroeconomic distribution of debt, η, responds to the rate of investment. A higher value of η_g would imply that the proportion of risky borrowers, η increase faster as the rate of investment increases. A higher η, on the other hand, by increasing the cumulative index of risk of default, Λ,

[8] Consider, for instance, the numerical values of the parameters used for illustration in the previous section. The configuration of parameters used therein satisfy the conditions mentioned above.

would tend to dampen the rate of investment. In other words, the macroeconomic mechanism which operates through η is essentially self-limiting and stabilizing. Hence, it complements the monetary authorities in stabilizing the system, reducing the necessity of stabilizing role of the rate of interest. A higher η_g would thus mean that the monetary authorities would, in general, need to keep the rate of interest lower in order to achieve the target rate, leading to an inverse relationship between η_g and the steady state rate of interest, $\bar{r}_7$.

Finally, we consider the effect of a small change in the parameter Λ_η on the steady state rate of interest, $\bar{r}_7$. Partially differentiating $\bar{r}_7$ with respect to Λ_η, we have

$$\frac{\partial \bar{r}_7}{\partial \Lambda_\eta} = -\frac{\hat{\mu}\eta_g\beta^2\sigma^2\psi^2 msu^{\star 2}}{\left(\hat{\mu}\Lambda_\lambda\beta k^2 s^2 + \alpha\beta\sigma^2\psi^2 m\right)u^\star + \hat{\mu}\Lambda_\lambda\sigma\psi k} < 0 \tag{39}$$

The parameter Λ_η, we recall, represents the sensitivity of the cumulative index of risk of default with respect to the proportion of risky borrowers in the macroeconomic distribution of debt. In other words, a higher value of Λ_η means placing a greater weightage of η in calculation of the cumulative index of risk of default. In other words, economies which are institutionally more sensitive to a worsening of borrower profile would have a higher Λ_η.[9] As would be evident, a higher Λ_η would mean a higher level of Λ for a given level of η, which in turn would, through the investment function, have a dampening effect on the rate of investment.

5.3 A Few Policy Considerations

We recall from our discussion in section 4 that local stability of the non-trivial steady state, E_7, depends on the rate of adjustment, l, of the rate of interest by the Central Bank. In particular, local stability requires that l is below certain critical value, $\hat{l}$. We next look at the relationship between this critical value, $\hat{l}$, and the level of capacity utilization, $u^\star$, desired by the Central Bank. To simplify this exercise, we consider a numerical example, where we keep the values of all the parameters except $u^\star$ same as those in section 4, i.e.

$$\begin{array}{lllllll} s = 0.3, & \sigma = 0.4, & \psi = 0.3, & \alpha = 0.5, & q = 0.6, & m = 0.6, & k = 0.7, \\ \beta = 0.8, & \bar{\mu} = 0.3, & \hat{\mu} = 0.4, & \eta_g = 0.1, & \Lambda_\eta = 0.1, & \Lambda_\lambda = 0.9, & \bar{\gamma} = 0.5 \end{array}$$

Using the computer algebra system, Maxima, we find that

$$\frac{\partial \hat{l}}{\partial u^\star} < 0 \ \forall \ u^\star \in]0, 1[\tag{40}$$

In other words, increasing the target level of capacity utilization by the Central Bank would imply either of the following:

1. Either this will tend to increase the possibility of making the economically meaningful steady state locally unstable, reducing the effectiveness of monetary policy and perhaps making it meaningless, or,

[9] For instance, the economies where financial crises and large-scale defaults are common might be more sensitive to such worsening of borrower profile than the economies where financial crises are of rare occurrence.

2. The Central Banks will be forced to cut back l, i.e. the rate of adjustment of the rate of interest to the rate of capacity utilization. Once again, this will have adverse implications for the effectiveness of the monetary policy.

Thus, the above discussion suggests a trade-off before the monetary authorities. Since targeting a rate of capacity utilization higher than a critical value might have adverse implications on the effectiveness of the monetary policy, the monetary authorities might be forced to keep down the desired rate of capacity utilization out of stability considerations, in order to keep the monetary policy effective. This, in turn, might conflict with the socio-political objectives of the policymakers, especially if stability considerations force them to target a rate of capacity utilization below the socially desired level.

For the same numerical exercise, we further note that

$$\frac{\partial \hat{l}}{\partial h} \in \,]0, \infty] \;\; \forall \; h \in \,]-\infty, \infty[\tag{41}$$

so that a higher h will imply a higher critical value of l, $\hat{l}$, representing the maximum speed with which the monetary authorities can adjust the rate of interest without creating stability problems. In other words, faster the private investors adjust the actual rate of investment to the desired rate, the more room the monetary authorities will have to conduct their monetary policy. This might be interpreted as a restriction imposed on the monetary authorities to coordinate their policy with the private sector. If the private sectors adjust their rate of investment faster, the monetary authorities can work within a wider range of l, and can also match the private sector in changing the rate of interest fast. On the other hand, if the private sector is slow in adjusting the actual rate of investment to the desired rate then the monetary authorities would also be forced to slow down their speed of adjustment of the rate of interest out of stability considerations.

One possible interpretation of the parameter, h, might be in terms of the degree of competition in the economy. One might argue, for instance, that in an economy with very little competition among firms, each firm will be under less pressure to adjust the actual rate of investment to the desired rate than a highly competitive economy, where competition would force firms to adjust their rate of investment faster. If we accept such an interpretation of h, then it would imply that the monetary authorities would have more room to conduct their monetary policy in a more competitive environment, since they will able to adjust the rate of interest faster. In other words, from the point of view of flexibility of monetary policy, an economy characterized by too little competition does not augur too well for the monetary authorities.

Finally, we should also note, from (40) and (41), that a higher level of h, interpreted above to be representing a more competitive economy, would also allow the monetary authorities to target a higher rate of capacity utilization, $u^{\star}$, without worrying about stability considerations. Further, as we noted above, since the economy grows in the steady state at the rate of capacity utilization desired by the central bank, we might conclude that a competitive economy might be characterized by a steady state rate of growth at higher rates of capacity utilization than a less competitive economy. In other words, absence of competition among firms will have adverse consequences in terms of the steady state rate of capacity utilization.

6 Bifurcations, Cycles and Complex Dynamics

Next, we turn to the case where the non-trivial steady state is not locally stable. As we mentioned above, a large literature on the class of two-predators one-prey Lotka-Volterra has emerged in recent times, which shows a rich variety of dynamic behaviors possible in such systems. In this section, we explore some of these possibilities.

6.1 Cyclical possibilities due to Hopf Bifurcation

We begin by exploring cyclical possibilities for the solution to the dynamical system represented by (30) from an application of Hopf Bifurcation theorem. We note the following:

Lemma 1. *For the dynamical system represented by* (30)*, under certain suitable parameter configuration, there exist a critical value, $\hat{l}$, of the parameter representing the rate of adjustment of the rate of interests by the monetary authorities, l, such that $l = \hat{l}$ provides a point of Hopf bifurcation.*

Proof. This follows from an application of Hopf Bifurcation Theorem in $\Re^{\mathbf{3}}$[10]. We find that $a_2 > 0$ provided $s > \max\left[\bar{\mu}, \frac{\sigma\psi}{k(q+r)}\right]$. We further note that the expression, $a_1 a_2 - a_3$ is a polynomial, which is linear in l (representing the rate of adjustment of the rate of interest by the monetary authorities) and quadratic in h (representing the rate of adjustment of the rate of investment by the private sector). By fixing one of the rates of adjustment, say h, it is possible to calculate a critical value of $l = \hat{l}$, such that the expression $a_1 a_2 - a_3$ vanishes. Hence, both the conditions required for existence of Hopf bifurcation theorem are satisfied.[11] □

For instance, if the parameters have the same value as in the example provided at the end of section 4, i.e.

$$\begin{array}{lllllll} s = 0.3, & \sigma = 0.4, & \psi = 0.3, & \alpha = 0.5, & q = 0.6, & m = 0.6, & k = 0.7, \\ \beta = 0.8, & \bar{\mu} = 0.3, & \hat{\mu} = 0.4, & \eta_g = 0.1, & \Lambda_\eta = 0.1, & \Lambda_\lambda = 0.9, & \bar{\gamma} = 0.5 \end{array}$$

[10] cf. Asada (1995), Asada & Semmler (1995), Flaschel (2009).

[11] It might be noted here that we are interpreting the term "Hopf bifurcation" in lemma 1 in a limited sense of emergence of purely imaginary roots. A stronger interpretation, in the sense of emergence of limit cycles would require fulfillment of non-degeneracy conditions. Further, one also needs to take into account the dependence of the first lyapunov coefficient on a second parameter, for instance, the rate of adjustment of the rate of investment by the private sector, h. Any conclusion regarding emergence of limit cycles from Hopf bifurcation must preclude destruction of these limit cycles due to the presence of a codim 2 bifurcation like the *Bautin bifurcation* [cf. Kuznetsov (1997, chapter 8, section 8.3), Guckenheimer & Kuznetsov (2007*a*)]. We leave an investigation into these concerns for future research.

then, with the monetary authorities targeting a capacity utilization of 80%, i.e. $u^\star = 0.8$, and the rate of adjustment of the rate of investment by the private sector fixed at $h = 0.8$, we have $a_2 \approx 0.5431l + .2266$ so that $a_2 > 0 \; \forall \; l \geq 0$; further, $a_1 a_2 - a_3 \approx 0.1192 - 0.3245l$, so that for a critical value of l, $\hat{l} \approx 0.3674$, the characteristic equation to the dynamical system represented by (30) will have purely imaginary roots.

It should be noted that existence of a Hopf bifurcation point due to appearance purely imaginary eigenvalues opens up possibilities of emergence of limit cycles, including stable limit cycles, as the parameter l passes through a critical value, $\hat{l}$. Existence of such stable limit cycles, which might be interpreted as growth cycles in g, r and d, might be verified by various methods, for instance, by following the method outlined in Kuznetsov (1997) and Edneral (2007), or by using any standard bifurcation software like XPPAUT or MATCONT.

Thus, with all other parameters fixed, as the monetary authorities try to adjust the rate of interest in an attempt to influence the capacity utilization, the parameter l might pass through its critical value, $\hat{l}$. In this case, a unique limit cycle might emerge in the neighborhood of $l = \hat{l}$. This limit cycle might be stable or unstable depending on parameter configuration; in case it is stable, then the solution trajectory is manifested as a growth cycle.

A few characteristics of this growth cycle should be pointed out here. To begin with, we notice that this is a three-dimensional cycle; in other words, the growth cycle involves cyclical fluctuations in all the three variables under consideration − the rate of investment or g, the rate of interest or r, and the debt-capital ratio or d. The fluctuations in these three variables, however, need not be synchronized in time. Further, there is a rather complicated set of macroeconomic feedback effects which accompany such a growth cycle.

Consider, for instance, a situation of upswing, characterized by an increase in g. As we noted in our discussion in section 3, this sets in a complex chain of events. We recall that an increase in g leads to an increase in capacity utilization, eventually having a positive impact on the g through an interaction of the multiplier and the accelerator. On its own, this would have been destabilizing and would have resulted in an unbounded system. This, however, is countered by a number of negative financial feedbacks.[12]

Firstly, during upswing, there is a credit expansion backed by a general optimism, resulting in credit being disbursed to those borrowers who would, in more pessimistic times, have been denied credit. This leads to a general deterioration of the profile of the borrowers, with an increase in the proportion of subprime borrowers in the macroeconomic distribution of debt, or an increase in η. This would, in turn, lead to an increase in the cumulative index of risk of default or Λ, which, eventually will dampen the rate of investment.

[12] As mentioned above, the macroeconomic feedback effects are analyzed by isolating each effect, ignoring the impact of other effects.

Secondly, during upswing, there is a greater recourse to debt to finance the increase in the rate of investment. This would have a positive impact on the stock of debt and debt-capital ratio or d. An increase in debt-capital ratio, in turn, would lead to an increase in the macroeconomic indicator of financial fragility or λ, and hence, the cumulative index of risk of default or Λ. This would, once again, eventually dampen the rate of investment.

Thirdly, during upswing, an increase in g will lead to an increase in the capacity utilization or u. As soon as the capacity utilization crosses its desired rate or $u^{\star}$, the monetary authorities will hike the rate of interest according to the interest rate rule. This will eventually dampen the rate of investment.

Finally, an increase in the rate of interest during the upswing, by the process mentioned above, will also lead to an increase the payment commitments, and hence, the debt-capital ratio or d. This might have two distinct effects. Firstly, it will increase the macroeconomic indicator of financial fragility or λ, eventually leading to an increase in the cumulative index of risk of default or Λ. Secondly, it will increase the borrowing requirements in order to pay the increased debt commitments, leading to an increase in the macroeconomic indicator of financial fragility or λ, increasing the cumulative index of risk of default and hence, rate of investment.

Each of the above feedback effects work with certain finite lag, so that the final trajectory is a complex outcome of several such feedbacks with diverse lags in opposite directions. We had seen in Datta (2011) and Datta (2012) that an interaction between some of these feedbacks are capable of producing cyclical outcomes. Results in this section show that all these feedbacks taken together are also capable of producing cyclical outcomes, albeit under certain specific configuration of parameter values.

A growth cycle, however, is only one of the possible outcomes for the system under consideration. All the above feedbacks taken together are also capable of producing several other dynamic outcomes. We turn to discussing some of them in the following section.

6.2 Other Bifurcations and Complex Dynamics

In the previous sections, we have seen that the dynamical system represented by (30) is capable of exhibiting both a convergence to steady state as well as cycles (due to Hopf bifurcation) around it. It must, however, be noted that the dynamical system being analyzed is capable of exhibiting much wider variety of dynamical behavior, including a wide variety of bifurcations, many of them leading to complex dynamics. While a complete and exhaustive discussion of such possibilities would be both beyond the scope of this paper as well as a digression from its central theme, in the following paragraphs we offer a brief discussion of a few broad directions which a future research in this area might explore.

We have already demonstrated the possiblity of emergence of limit cycles through Hopf bifurcation in our model in section 6.1 above. An obvious extension of this exercise would be to formulate a Poincaré map for these limit cycles. An investigation of the multipliers associated with the Jacobian matrix of the Poincaré map would reveal possibilities of a *Neimark-Sacker bifurcation* leading to emergence of invariant torus, in the neighborhood of the point where these multipliers become purely imaginary. Further, a preliminary investigation of the coefficients of the characteristic equation of the jacobian matrix evaluated at E_7 (using the computer algebra system, Maxima) reveals the following:

1. We note that $l = 0 \;\Rightarrow\; a_3 = 0$, i.e. if the monetary authorities follow a completely passive monetary policy, then the product of the three eigenvalues, ϖ_1, ϖ_2 and ϖ_3, to the jacobian matrix at E_7, vanishes. This is possible if and only if at least one of the eigenvalues is zero.

2. If the above happens while $a_1, a_2 > 0$, then we have a *fold bifurcation* at $l = 0$, also known as a *saddle-node bifurcation.* A variety of interesting dynamics might emerge in the neighborhood of such a bifurcation for a three-dimensional system of differential equations, including generation of multiple homoclinic orbits, disappearance of saddle-nodes through *Shil'nikov bifurcation* leading to complex dynamics due to generation of an infinite number of saddle-periodic orbits. [cf. Kuznetsov (1997, chapter 3 & 6), Kuznetsov (2006)]

3. However, we also note that a_1 is a linear function of h. For a critical value, $\hat{h}$, of h, where
$$\hat{h} = \frac{m\,(q + \bar{r}_7)\,\bar{d}_7}{(\bar{\mu} - s\beta - \alpha\bar{r}_7 - 2\hat{\mu}\Lambda_\eta\eta_g s\beta u^\star)\,u^\star}$$
we have $a_1 = 0$. If this happens along with $l = 0 \;\Rightarrow\; a_3 = 0$, while $a_2 > 0$, i.e. both the sum as well as product of the roots are zero, while sum of the product of the roots taken two at a time is positive, then it follows that we have a zero eigenvalue along with a pair of purely imaginary eigenvalues. This represents a case of *Gavrilov-Guckenheimer bifurcation*, also known as *Fold-Hopf bifurcation.* This might trigger a variety of complex dynamics, including bifurcation of Shil'nikov homoclinic and heteroclinic orbits, appearance of invariant torus around a Hopf bifurcation limit cycle and its subsequent breakdown leading to chaotic dynamics. [cf. Kuznetsov (1997, chapter 8, section 8.5), Guckenheimer & Kuznetsov (2007*c*)]

4. Further, we note that $h = 0 \Rightarrow a_2 = 0$ & $a_1 > 0$, i.e. if the private sector is completely passive in adjusting the rate of investment,[13] then the sum of product of eigenvalues taken two at a time vanishes. If this happens along with a passive monetary policy, i.e. $l = 0 \Rightarrow a_3 = 0$, then we have a situation of zero eigenvalue with a multiplicity of two. This represents a case of *Bogdanov-Takens bifurcation*, also known as *Double-zero*

[13] This might be the case, for instance, in an economy characterized by too little competition, where firms have little incentive to adjust their rate of investment based on information on current values of variables.

bifurcation, leading to a variety of interesting dynamical behaviors in the neighborhood of such a bifurcation.[cf. Kuznetsov (1997, chapter 8, section 8.4), Guckenheimer & Kuznetsov (2007*b*)]

It may be mentioned here that many of the dynamics mentioned above might be observed from a numerical simulation exercise, using any standard bifurcation software like XPPAUT or MATCONT.

While the above discussion is not meant to be an exhaustive list of possible dynamics emerging around the non-trivial steady state, E_7 for the dynamical system represented by (30), we can draw a few general conclusions:

1. Depending on the configuration of the parameters, the dynamical system represented by (30) is capable of exhibiting a wide variety of dynamics, ranging from simple convergence or cycles to very complicated dynamics including chaotic dynamics. Hence, it follows that, without a high degree of accuracy in measurement of various parameters, our model is likely to have limited capability in predicting the behavior of the solution trajectory. Since such high degree of accuracy in measuring economic parameters is usually not expected, most of the discussion in this paper must be looked upon in terms of possibilities rather than actual prediction of the solution trajectory.

2. One might also note the important role played by the monetary authorities in stabilizing the system. As we noted above, a low value of the rate of adjustment, l, of rate of interest by the monetary authorities opens up the possibilities of a wide variety of complicated and chaotic dynamics. A low value of l corresponds to a failure by the monetary authorities to effectively control the rate of interest. Such a situation might arise, for instance, in the absence of a one-to-one correspondence between the bank rate controlled by the Central Bank and the market lending rate offered by the financial intermediaries. A full discussion of such a scenario, however, would necessitate a more complete model that takes into account factors like inflation and international capital flows leading to such a breakdown of linkage between bank rate and the market lending rate, and hence, is beyond the scope of the present work. We leave this exercise as an area of future research.

7 Concluding Remarks

The main conclusions we can draw from our discussion in the preceding sections are as follows:

1. We note, first of all, that there are a large number of macroeconomic feedback effects work simultaneously at work in these models. The final outcome, which is the aggregate of all the macroeconomic feedback effects, would depend on the relative strengths of these effects as well as the time taken to complete each of them. Hence, a model

of this kind is capable of exhibiting diverse dynamical possibilities. We, however, primarily focus our attention on two such possibilities of interest: convergence and cyclical possibilities. We find that under certain suitable configuration of parameters, there are possibilities for both.

2. We also find that under certain conditions, the financial factors discussed in this paper provide us with endogenous ceilings and floors to our system. For instance, in case of convergence to E_7, since E_7 is the only steady state in the positive real number space, the model is bounded. Similarly, in case of a limit cycle emerging from Hopf bifurcation (i.e. when the Hopf bifurcation point is non-degenerate), irrespective of whether the Hopf bifurcation is supercritical or subcritical, the limit cycle will enclose a bounded region within which the dynamics will be restricted. In other words, under certain configuration of parameters, the solution is bounded. We have already shown numerical illustrations of such configurations of parameters.

3. We further note that in the long-run steady state, the economy grows at a rate of capacity utilization desired by the central bank. In other words, as long as the dynamics are bounded in the short-run, and our model does not break down, the monetary policy is effective in determining the rate of capacity utilization in the long-run steady-state. However, there are several question-marks on the effectiveness of the monetary policy. Firstly, in case the solution trajectory does not converge to the steady state, we have a situation where the desired capacity utilization might not be attained in the short-run. This might be the case, for instance, if the trajectory cycles around the steady state in the short-run, or moves about chaotically within a bounded region without ever attaining the steady state. In this case, therefore, the effectiveness of monetary policy in the form of an interest rate rule might be questioned. Secondly, as we noted in our discussion above, at times the monetary authorities might be forced to target a rate of capacity utilization which is socially not desirable, in order to ensure stability of the system.[14] Under such a situation, the rate of capacity utilization actually achieved by the monetary authorities might differ from what the policymakers might have wanted to target.

4. Finally, we should also point out that the cyclical conclusions emerging from Hopf bifurcation must be subjected to further tests of non-degeneracy and stability of limit cycles. We leave an investigation of these issues for future research.

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[14]See, for instance, the condition imposed by (40); targeting a higher level of capacity utilization reduces some of the flexibility available to the monetary authorities.

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